AVERAGE-CASE PERTURBATIONS AND SMOOTH CONDITION NUMBERS

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Abstract. In this paper we define a new condition number adapted to directionally uniform perturbations. The definitions and theorems can be applied to a large class of problems. We show the relation with the classical condition number, and study some interesting examples.

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1. Introduction

Let X and Y be two real (or complex) Riemannian manifolds of real dimensions m and n $(m \geq n)$ respectively associated to some computational problem, where X is the space of *inputs* and Y is the space of *outputs*. Let $V \subset X \times Y$ be the *solution variety*, i.e. the subset of pairs (x, y) such that y is an output corresponding to the input x. Let $\pi_1 : V \to X$ and $\pi_2 : V \to Y$ be the canonical projections, and $\Sigma \subset V$ be the set of critical points of the projection π_1.

In case $\dim V = \dim X$, for each $(x, y) \in V \setminus \Sigma$ there is a differentiable function locally defined from some neighborhoods U_x and U_y of $x \in X$ and $y \in Y$ respectively, namely

$$G := \pi_2 \circ \pi_1^{-1}|_{U_x} : U_x \to U_y.$$

Let us denote by $\langle \cdot, \cdot \rangle_x$ and $\langle \cdot, \cdot \rangle_y$ the Riemannian (or Hermitian) inner product in the tangent spaces T_xX and T_yY at x and y respectively. The derivative $DG(x) : T_xX \to T_yY$ is called the *condition matrix* at (x, y). The *classical condition number* at $(x, y) \in V \setminus \Sigma$ is defined as

$$\kappa(x, y) := \max_{\substack{\dot{x} \in T_xX \\ \|\dot{x}\|^2{}_x = 1}} \|DG(x)\dot{x}\|_y. \tag{1}$$

This number is an upper-bound -to first-order approximation- of the *worst-case* sensitivity of the output error with respect to small perturbations of the input. It plays an important role to understand the behavior of algorithms and, as a consequence, appears in the usual bounds of the running time of execution. There exist an extensive literature about the role of the condition

number in numerical analysis and complexity of algorithms, see for example [13] and references therein.

In many practical situations, there exist a discrepancy between theoretical analysis and observed performance of an algorithm. There exist several approaches that attempt to rectify this discrepancy. Among them we find *average-case analysis* (see [7, 10]) and *smooth analysis* (see [11, 4, 15]). For a comprehensive review on this subject with historical notes see [5].

In this paper, averaging is performed in a different form. In many problems, the space of inputs has a much larger dimension than the one of the space of outputs ($m \gg n$). Then, it is natural to assume that infinitesimal perturbations of the input will produce drastic changes in the output, only when they are performed in a few directions. Then, a possibly different approach to analyze complexity of algorithms is to replace "worst direction" by a certain mean over all possible directions. This alternative was already suggested and studied in [14] in the case of linear system solving $Ax = b$, and more generally, in [12] in the case of matrix perturbation theory where the first-order perturbation expansion is assumed to be random.

In this paper we extend this approach to a large class of computational problems, restricting ourselves to the case of directionally uniform perturbations.

Generalizing the concept introduced in [14] and [12], we define the *pth-average condition number* at (x, y) as

$$\kappa_{av}{}^{[p]}(x,y) := \left[\frac{1}{\mathrm{vol}(S_x^{m-1})}\int_{\dot{x}\in S_x^{m-1}} \|DG(x)\dot{x}\|_y^p \, dS_x^{m-1}(\dot{x})\right]^{1/p} \qquad (p = 1, 2, \ldots) \tag{2}$$

where $\mathrm{vol}(S_x^{m-1}) = \frac{2\pi^{m/2}}{\Gamma(m/2)}$ is the measure of the unit sphere S_x^{m-1} in T_xX, and dS_x^{m-1} is the induced volume element. We will be mostly interested in the case $p = 2$, which we simply write κ_{av} and call it *average condition number.*

Before the statement of the main theorem, we define the *Frobenius condition number* as

$$\kappa_F(x,y) := \|DG(x)\|_F = \sqrt{\sigma_1^2 + \cdots + \sigma_n^2}$$

where $\|\cdot\|_F$ is the Frobenius norm and $\sigma_1, \ldots, \sigma_n$ are the singular values of the condition matrix.

Theorem 1.

$$\kappa_{av}{}^{[p]}(x,y) = \frac{1}{\sqrt{2}}\left[\frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m+p}{2}\right)}\right]^{1/p} \mathbb{E}(\|\eta_{\sigma_1,\ldots,\sigma_n}\|^p)^{1/p}$$

where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$ and $\eta_{\sigma_1,\ldots,\sigma_n}$ is a centered Gaussian vector in $\mathbb{R}^n$ with diagonal covariance matrix $Diag(\sigma_1^2, \ldots, \sigma_n^2)$.
In particular,

$$\kappa_{av}(x,y) = \frac{\kappa_F(x,y)}{\sqrt{m}}. \tag{3}$$

This result is most interesting when $n \ll m$, for in that case

$$\kappa_{av}(x,y) \le \sqrt{\frac{n}{m}} \cdot \kappa(x,y) \ll \kappa(x,y),$$

as $\kappa_F(x,y) \leq \sqrt{n}\cdot\kappa(x,y)$. Thus, in these cases one may expect much better stability properties than those predicted by classical condition numbers.

In numerical analysis, many authors are interested in relative errors. Thus, in the case $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are normed vector spaces, instead of consider the (absolute) condition number (1), one could take the *relative condition number* defined as

$$\kappa_{rel}(x) := \frac{\|x\|_X}{\|y\|_Y}\kappa(x,y),$$

and the *relative Frobenius condition number* as

$$\kappa_{relF}(x) := \frac{\|x\|_X}{\|y\|_Y}\kappa_F(x,y),$$

(for simplicity we drop the y in the argument).

In the same way, we define the *relative pth-average condition number* as

$$\kappa_{rel}{}_{av}^{[p]}(x) := \frac{\|x\|_X}{\|y\|_Y}\kappa_{av}{}^{[p]}(x,y), \quad (p = 1, 2, \ldots). \tag{4}$$

For the case $p = 2$ we simply write κ_{relav} and call it *relative average condition number.*

Theorem 1 remains true if one change the (absolute) condition number by the relative condition number. In particular,

$$\kappa_{relav}(x) := \frac{\kappa_{relF}(x,y)}{\sqrt{m}}.$$

2. Componentwise Analysis

In the case $Y = \mathbb{R}^n$ we define the *kth-componentwise condition number* at $(x, y) \in V$ as:

$$\kappa(x,y,k) := \max_{\substack{\dot{x}\in T_xX \\ \|\dot{x}\|^2{}_x=1}} |(DG(x)\dot{x})_k| \qquad (k = 1,\ldots,n), \tag{5}$$

where $|\cdot|$ is the absolute value and w_k indicates the kth-component of the vector $w \in \mathbb{R}^n$.

Following [14] for the linear case, we define the average componentwise condition number as

$$\kappa_{av}{}^{[p]}(x,y,k) := \left[\frac{1}{\mathrm{vol}(S_x^{m-1})}\int_{\dot{x}\in S_x^{m-1}} |(DG(x)\dot{x})_k|^p \; dS_x^{m-1}(\dot{x})\right]^{1/p} \quad (p = 1, 2, \ldots). \tag{6}$$

Then we have:

Proposition 1.

$$\kappa_{av}{}^{[p]}(x,y,k) = \left[\frac{1}{\sqrt{\pi}}\cdot\frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m+p}{2}\right)}\cdot\Gamma\left(\frac{p+1}{2}\right)\right]^{1/p}\cdot\kappa_F(x,y,k).$$

In particular,

$$\kappa_{av}(x,y,k) = \frac{\kappa_F(x,y,k)}{\sqrt{m}}.$$

Proof. Observe that ${\kappa_{av}}^{[p]}(x,y,k)$ is the pth-average condition number for the problem of finding the kth-component of $G=(G_1,\ldots,G_n)$. *Theorem 1* applied to G_k yields

$$ {\kappa_{av}}^{[p]}(x,y,k)=\frac{1}{\sqrt{2}}\left[\frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m+p}{2}\right)}\right]^{\frac{1}{p}}\mathbb{E}(|\eta_{\sigma_1}|^p)^{1/p} $$

where $\sigma_1=\|DG_k(x)\|=\kappa(x,y,k)$. Then,

$$ \mathbb{E}(|\eta_{\sigma_1}|^p)^{1/p}=\kappa(x,y,k)\cdot\mathbb{E}(|\eta_1|^p)^{1/p} $$

where η_1 is a standard normal in $\mathbb{R}$. Finally,

$$ \mathbb{E}(|\eta_1|^p)=\frac{2}{\sqrt{2\pi}}\int_0^\infty \rho^p e^{-\rho^2/2}\,d\rho=\frac{2}{\sqrt{2\pi}}2^{\frac{p-1}{2}}\Gamma(\frac{p+1}{2}) $$

and the proposition follows. □

3. Examples

In this section we will compute the average condition number for different problems: systems of linear equations, eigenvalue and eigenvector problems, finding kernels of linear transformations and solving polynomial systems of equations. The first two have been computed in [12] and are an easy consequence of *Theorem 1* and the usual condition number.

3.1. **Systems of Linear Equations.**

We consider the problem of solving the system of linear equations $Ay=b$, where $A\in\mathscr{M}_n(\mathbb{R})$ the space of $n\times n$ matrices with the Frobenius inner product, i.e. $\langle A,B\rangle_F=\mathrm{trace}(B^tA)$ (B^t is the transpose of B), and $b\in\mathbb{R}^n$.
If we assume that b is fixed, then the input space $X=\mathscr{M}_n(\mathbb{R})$ with the Frobenius inner product, $Y=\mathbb{R}^n$ with the Euclidean inner product, and Σ equals the subset of non-invertible matrices. Then the map $G:\mathscr{M}_n(\mathbb{R})\setminus\Sigma\to\mathbb{R}^n$ is globally defined and differentiable, namely

$$ G(A)=A^{-1}b\ (=y). $$

By implicit differentiation,

$$ DG(A)(\dot{A})=-A^{-1}\dot{A}y. \tag{7} $$

Is easy to see from (7) that $\kappa(A,y)=\|A^{-1}\|\cdot\|y\|$ and $\kappa_F(A,y)=\|A^{-1}\|_F\cdot\|y\|$. Then from *Theorem 1* we get

$$ \kappa_{av}(A,y)=\frac{\|A^{-1}\|_F\cdot\|y\|}{n}\leq\frac{\kappa(A,y)}{\sqrt{n}}. \tag{8} $$

A similar result was proved in [12].

For the general case, we have $X=\mathscr{M}_n(\mathbb{R})\times\mathbb{R}^n$ with the Frobenius inner product in $\mathscr{M}_n(\mathbb{R})$ and the Euclidean inner product in $\mathbb{R}^n$. Then, $G:\mathscr{M}_n(\mathbb{R})\setminus\Sigma\times\mathbb{R}^n\to\mathbb{R}^n$ satisfies $G(A,b)=A^{-1}b$. Is easy to see that $\kappa((A,b),y)=\|A^{-1}\|\cdot\sqrt{1+\|y\|^2}$ and $\kappa_F((A,b),y)=\|A^{-1}\|_F\cdot\sqrt{1+\|y\|^2}$. Again from *Theorem 1* we get

$$ \kappa_{av}((A,b),y)=\frac{\|A^{-1}\|_F\cdot\sqrt{1+\|y\|^2}}{\sqrt{n^2+n}}\leq\frac{\kappa((A,b),y)}{\sqrt{n+1}}. $$

For the kth-componentwise condition number, we have that

$$\kappa_{av}((A,b),y,k) = \left[\frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma\left(\frac{n^2}{2}\right)}{\Gamma\left(\frac{n^2+p}{2}\right)} \cdot \Gamma\left(\frac{p+1}{2}\right)\right]^{1/p} \cdot \kappa((A,b),y).$$

A similar result was proved in [14].

In [7], it is proved that the expected value of the relative condition number $\kappa_{rel}(A) = \|A\| \cdot \|A^{-1}\|$ of a random matrix A whose elements are i.i.d standard normal, satisfies:

$$\mathbb{E}(\log \kappa_{rel}(A)) = \log n + c + o(1),$$

as $m \to \infty$, where $c \approx 1.537$. If we consider the relative average condition number defined in (4), we get from (8)

$$\mathbb{E}(\log \kappa_{relav}(A)) = \frac{1}{2}\log n + c + o(1),$$

as $m \to \infty$.

3.2. **Eigenvalue and Eigenvector Problem.** Let $X = \mathscr{M}_n(\mathbb{C})$ be the space of $n \times n$ complex matrices with the Frobenius Hermitian inner product, $Y = \mathbb{P}(\mathbb{C}^{n+1}) \times \mathbb{C}$ and $V = \{(A, v, \lambda) : Av = \lambda v\}$. Then for $(A, v, \lambda) \in V \setminus \Sigma$ the condition matrices DG_1 and DG_2 associated with the eigenvector and eigenvalue problem are

$$DG_1(A)\dot{A} = (\pi_{v^\perp}(\lambda I - A)|_{v^\perp})^{-1}\left(\pi_{v^\perp}\dot{A}v\right) \quad \text{and} \quad DG_2(A)\dot{A} = \frac{\langle u, \dot{A}v\rangle}{\langle u, v\rangle},$$

where u is some left eigenvector associated with λ, i.e. $u^*A = \overline{\lambda}u^*$ (see [Bez IV]). The associated condition numbers are:

$$\kappa_1(A,v) = \left\|(\pi_{v^\perp}(\lambda I - A)|_{v^\perp})^{-1}\right\| \quad \text{and} \quad \kappa_2(A,\lambda) = \frac{\|u\| \cdot \|v\|}{|\langle u, v\rangle|}. \tag{9}$$

From our *Theorem 1*, we get the respective average condition numbers:

$$\kappa_{av1}(A,v) = \frac{1}{n}\left\|(\pi_{v^\perp}(\lambda I - A)|_{v^\perp})^{-1}\right\|_F \le \frac{1}{\sqrt{n}}\kappa_1(A,v),$$

$$\kappa_{av2}(A,\lambda) = \frac{1}{n}\kappa_2(A,\lambda).$$

A similar result for $\kappa_{av2}(A,\lambda)$ was proved in [12].

3.3. **Finding Kernels of Linear Transformations.** Let $\mathscr{M}_{k,p}(\mathbb{C})$ be the linear space of $k \times p$ complex matrices with the Frobenius Hermitian inner product, i.e. $\langle A, B\rangle_F = \text{trace}(B^*A)$ (B^* is the adjoint of B), and $\mathcal{R}_r \subset \mathscr{M}_{k,p}(\mathbb{C})$ the subset of matrices of rank r. We consider the problem of solving the system of linear equations $Ax = 0$. For this purpose, we introduce the *Grassmannian* $\mathbb{G}_{p,\ell}$ of complex subspaces of dimension ℓ in $\mathbb{C}^p$, where $\ell = \dim(\ker A) = p - r$.

The input space $X = \mathcal{R}_r$ is a smooth submanifold of $\mathscr{M}_{k,p}(\mathbb{C})$ of complex dimension $(k+p)r - r^2$ (see [6]). Thus, it has a natural Hermitian structure induced by the Frobenius metric on $\mathscr{M}_{k,p}(\mathbb{C})$.

In what follows, we identify $\mathbb{G}_{p,\ell}$ with the quotient $\mathbb{S}_{p,\ell}/\mathcal{U}_\ell$ of the *Stiefel* manifold

$$\mathbb{S}_{p,\ell} := \{M \in \mathscr{M}_{p,\ell}(\mathbb{C}) : M^*M = I\}$$

by the unitary group $\mathcal{U}_\ell \subset \mathscr{M}_\ell(\mathbb{C})$, which acts on the right of $\mathbb{S}_{p,\ell}$ in the natural way (see [6]). Then, the complex dimension of the output space $Y = \mathbb{G}_{p,\ell}$ is $(p-r)r$.

We will use the same symbol to represent an element of $\mathbb{S}_{p,\ell}$ and its class in $\mathbb{G}_{p,\ell}$. The manifold $\mathbb{S}_{p,\ell}$ has a canonical Hermitian structure induced by the Frobenius norm in $\mathscr{M}_\ell(\mathbb{C})$. On the other hand, $\mathcal{U}_\ell$ is a Lie group of isometries acting on $\mathbb{S}_{p,\ell}$. Therefore, $\mathbb{G}_{p,\ell}$ is a Homogeneous space (see [8]), with a natural Riemannian structure that makes the projection $\mathbb{S}_{p,\ell} \to \mathbb{G}_{p,\ell}$ a Riemannian submersion. The orbit of $M \in \mathbb{S}_{p,\ell}$ under the action of the unitary group $\mathcal{U}_\ell$, namely, $\mathrm{o}_\ell(M) = \{MU : U \in \mathcal{U}_\ell\}$, defines a smooth submanifold of $\mathbb{S}_{p,\ell}$. In this form we can define a local chart of a small neighborhood of $M \in \mathbb{G}_{p,\ell}$ from the affine spaces

$$M + T_M \mathrm{o}_\ell(M)^\perp$$

where $T_M\mathrm{o}_\ell(M)^\perp$ is the orthogonal complement of $T_M\mathrm{o}_\ell(M)$ in $T_M\mathbb{S}_{p,\ell}$. Implicit differentiation in local coordinates of the input-output map G at the point $G(A) = M \in \mathbb{S}_{p,\ell}$ yields

$$DG(A)(\dot{A}) = -A^\dagger \dot{A} M, \tag{10}$$

where $\dot{A} \in T_A\mathcal{R}_r$, $A^\dagger$ is the Moore-Penrose inverse of A and $A^\dagger\dot{A}M \in T_M\mathrm{o}_\ell(M)^\perp$.

One way to compute the singular values of the condition matrix described in (10), is to take an orthonormal basis in $\mathscr{M}_{k,p}(\mathbb{C})$ which diagonalizes A. From the singular value decomposition, there exists orthonormal basis $\{u_1, \ldots, u_k\}$ of $\mathbb{C}^k$, and $\{v_1, \ldots, v_p\}$ of $\mathbb{C}^p$, such that $Av_i = \sigma_i u_i$ for $i = 1, \ldots, r$, and $Av_i = 0$ for $i = r+1, \ldots, p$. Thus, $\{u_i v_j^* : i = 1, \ldots, k;\ j = 1, \ldots, p\}$ is an orthonormal basis of $\mathscr{M}_{k,p}(\mathbb{C})$ which diagonalizes A. In this basis the tangent space $T_A\mathcal{R}_r$ is the orthogonal complement of the subspace generated by $\{u_i v_j^* : i = r+1, \ldots, k;\ j = r+1, \ldots, p\}$. From where we conclude that

$$\kappa(A, M) = \|DG(A)\| = \|A^\dagger\|, \quad \kappa_F(A, M) = \sqrt{p-r} \cdot \|A^\dagger\|_F.$$

From our *Theorem 1*,

$$\kappa_{av}(A, M) = \frac{\sqrt{p-r}}{\sqrt{(k+p-r)r}} \cdot \|A^\dagger\|_F \leq \sqrt{\frac{p(p-r)}{(k+p-r)r}} \cdot \kappa(A, M).$$

In [1], it is proved that

$$\mathbb{E}(\log \kappa_{rel}(A) : A \in \mathcal{R}_r) \leq \log\left[\frac{k+p-r}{k+p-2r+1}\right] + 2.6,$$

where the expected value is computed with respect to the normalized naturally induced measure in $\mathcal{R}_r$. Our *Theorem 1* immediately yields a bound for the average relative condition number, namely,

$$\mathbb{E}(\log \kappa_{relav}(A) : A \in \mathcal{R}_r) \leq \frac{1}{2}\log\left[\frac{(k+p-r)r}{(k+p-2r+1)^2 p(p-r)}\right] + 2.6.$$

3.4. **Finding Roots Problem I: Univariate Polynomials.** We start with the case of one polynomial in one complex variable. Let $X = \mathcal{P}_d = \{f : f(z) = \sum_{i=0}^{d} f_i z^i,\ f_i \in \mathbb{C}\}$. Identifying $\mathcal{P}_d$ with $\mathbb{C}^{d+1}$ we can define two standard inner products in the space $\mathcal{P}_d$:

- Weyl inner product:

$$\langle f, g\rangle_W := \sum_{i=0}^{d} f_i \overline{g_i} \binom{d}{i}^{-1}; \tag{11}$$

- Canonical Hermitian inner product:

$$\langle f, g\rangle_{\mathbb{C}^{d+1}} := \sum_{i=0}^{d} f_i \overline{g_i}. \tag{12}$$

The solution variety is given by $V = \{(f,z) : f(z) = 0\}$. Thus, by implicit differentiation

$$DG(f)(\dot{f}) = -\left(f'(\zeta)\right)^{-1} \dot{f}(\zeta).$$

We denote by κ_W and $\kappa_{\mathbb{C}^{d+1}}$ the condition numbers with respect to the Weyl and Euclidean inner product. The reader may check that

$$\kappa_W(f,\zeta) = \frac{(1+|\zeta|^2)^{d/2}}{|f'(\zeta)|} \quad \text{and} \quad \kappa_{\mathbb{C}^{d+1}}(f,\zeta) = \frac{\sqrt{\sum_{i=0}^{d} |\zeta|^{2i}}}{|f'(\zeta)|},$$

(for a proof see [3], p. 228). From *Theorem 1*, we get:

$$\kappa_{av}{}^{[2]}_{W}(f,\zeta) = \frac{1}{\sqrt{2(d+1)}} \kappa_W(f,\zeta), \qquad \kappa_{av}{}^{[2]}_{\mathbb{C}^{d+1}}(f,\zeta) = \frac{1}{\sqrt{2(d+1)}} \kappa_{\mathbb{C}^{d+1}}(f,\zeta).$$

3.5. **Finding Roots Problem II: Systems of Polynomial Equations.** We now study the case of complex homogeneous polynomial systems. Let $\mathcal{H}_{(d)}$ the space of systems $f : \mathbb{C}^{n+1} \to \mathbb{C}^n$, $f = (f_1, \ldots, f_n)$ where each f_i is a homogenous polynomial of degree d_i. We consider $\mathcal{H}_{(d)}$ with the homogeneous analogous of the Weyl structure defined above (see Chapter 12 of [3] for details).

Let $X = \mathbb{P}(\mathcal{H}_{(d)})$ and $Y = \mathbb{P}(\mathbb{C}^{n+1})$ and $V = \{(f,\zeta) : f(\zeta) = 0\}$. We denote by $N = \sum_{i=1}^{n} \binom{d_i+n}{n} - 1$ the complex dimension of X. We may think of $2N$ as the size of the input.

Then,

$$DG(f)(\dot{f}) = -\left(Df(\zeta)_{\zeta^\perp}\right)^{-1} \dot{f}(\zeta),$$

and the condition number is

$$\kappa_W(f,\zeta) = \left\| \left(Df(\zeta)_{\zeta^\perp}\right)^{-1} \right\|,$$

where some norm 1 affine representatives of f and ζ have been chosen (cf.[3]). Associated with this quantity, we consider

$$\kappa_W(f) := \sqrt{\frac{1}{\mathcal{D}} \sum_{\{\zeta : f(\zeta)=0\}} \kappa_W(f,\zeta)^2}, \tag{13}$$

where $\mathcal{D} = d_1 \cdots d_n$ is the number of projective solutions of a generic system.

The expected value of $\kappa_W^2(f)$ is an essential ingredient in the complexity analysis of path-following methods (cf. [9], [2]). In [2] the authors proved that

$$\mathbb{E}_f \left[\kappa_W(f))^2\right] \leq 8nN, \tag{14}$$

where f is chosen at random with the Weyl distribution.

The relation between complexity theory and κ_{av} is not clear yet. However, it is interesting to study the expected value of the κ_{av}-analogous of equation (14), namely

$$\kappa_{av\,W}(f) := \sqrt{\frac{1}{\mathcal{D}} \sum_{\{\zeta :\, f(\zeta)=0\}} \kappa_{av\,W}(f,\zeta)^2}.$$

From our *Theorem 1* we get,

$$\kappa_{av\,W}(f,\zeta) \leq \frac{\kappa_W(f,\zeta)}{\sqrt{N/n}}, \qquad \mathbb{E}_f\left[\kappa_{av\,W}(f))^2\right] \leq 8\,n^2.$$

Note that the last bound depends on the number of unknowns n, and not on the size of the input $n \ll N$.

4. Proof of the main Theorem

In the case of complex manifolds, the condition matrix turns to be an $n \times n$ complex matrix. In what follows, we identify it with the associated $2n \times 2n$ real matrix. We center our attention in the real case.

The main theorem follows immediately from *Lemma 1* and *Proposition 2* below.

Lemma 1. *Let η be a Gaussian standard random vector in $\mathbb{R}^m$. Then*

$$\kappa_{av}{}^{[p]}(x,y) = \frac{1}{\sqrt{2}}\left[\frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m+p}{2}\right)}\right]^{\frac{1}{p}} \cdot \left[\mathbb{E}(\|DG(x)\eta\|^p)\right]^{1/p},$$

where $\mathbb{E}$ is the expectation operator and $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$.

Proof. Let $f : \mathbb{R}^m \to \mathbb{R}$ be the continuous function given by

$$f(v) = \|DG(x)v\|.$$

Then

$$\left[\mathbb{E}(\|DG(x)\eta\|^p)\right]^{1/p} = \left[\frac{1}{(2\pi)^{m/2}}\int_{\mathbb{R}^m} f(v)^p \cdot e^{-\|v\|^2/2}\, dv\right]^{1/p}.$$

Integrating in polar coordinates we get that

$$\mathbb{E}(\|DG(x)\eta\|^p) = \frac{I_{m+p-1}}{(2\pi)^{m/2}} \cdot \int_{S^{m-1}} f^p \, dS^{m-1} \tag{15}$$

where

$$I_j = \int_0^{+\infty} \rho^j\, e^{-\rho^2/2}\, d\rho, \quad j \in \mathbb{N}.$$

Making the change of variable $u = \rho^2/2$ we obtain

$$I_j = 2^{\frac{j-1}{2}}\Gamma(\frac{j+1}{2}),$$

therefore

$$I_{m+p-1} = 2^{\frac{m+p-2}{2}} \cdot \Gamma\left(\frac{m+p}{2}\right). \tag{16}$$

Then joining together (15) and (16) we obtain the result. □

Proposition 2.

$$\mathbb{E}(\|DG(x)\eta\|^p) = \mathbb{E}(\|\eta_{\sigma_1,\ldots,\sigma_n}\|^p)$$

where $\eta_{\sigma_1,\ldots,\sigma_n}$ is a centered Gaussian vector in $\mathbb{R}^n$ with diagonal covariance matrix $Diag(\sigma_1^2,\ldots,\sigma_n^2)$.

Proof. Let $DG(x) = UDV$ be a singular value decomposition of $DG(x)$. By the invariance of the Gaussian distribution under the action of the orthogonal group in X, $V\eta$ is again a standard Gaussian random vector. Then,

$$\mathbb{E}(\|DG(x)\eta\|^p) = \mathbb{E}(\|UD\eta\|^p),$$

and by the invariance under the action of the orthogonal group of the Euclidean norm, we get

$$\mathbb{E}(\|DG(x)\eta\|^p) = \mathbb{E}(\|D\eta\|^p).$$

Finally $D\eta$ is a centered Gaussian vector in $\mathbb{R}^n$ with covariance matrix $Diag(\sigma_1^2,\ldots,\sigma_n^2)$, and the proposition follows. For the case $p = 2$

$$\kappa_{av}(x,y) = \left[\mathbb{E}\left(\sigma_1^2\eta_1^2 + \ldots + \sigma_n^2\eta_n^2\right)\right]^{1/2}$$

where $\eta_1,\ldots,\eta_n$ are i.i.d. standard normal in $\mathbb{R}$. Then

$$\kappa_{av}(x,y) = \left(\sum_{i=1}^{n}\sigma_i^2\right)^{1/2} = \kappa_F(x,y).$$

□

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